THE GROUPS OF STEINER IN PROBLEMS OF CONTACT

(SECOND PAPER)*

BY

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1. Denote by G the group of the equation upon which depends the determination of the curves of order $n-3$ having simple contact at $\frac{1}{2}n(n-3)$ points with a given curve C_n of order n having no double points. The case in which n is odd was discussed in the former paper (Transactions, January, 1902) and G was shown to be a subgroup of the group defined by the invariants $\phi_3, \phi_4, \phi_5, \cdots$, the latter group being holoedrically isomorphic with the first hypoabelian group on $2p$ indices with coefficients taken modulo 2. For n even, G is contained in the group H defined by the invariants $\phi_4, \phi_6, \cdots$, with even subscripts. JORDAN has shown (*Traité*, pp. 229–242) that H is holoedrically isomorphic with the abelian linear group A on $2p$ indices with coefficients taken modulo 2. The object of the present paper is to establish the latter theorem by a short, elementary proof, which makes no use of the abstract substitutions $[\alpha_1, \beta_1, \cdots, \alpha_p, \beta_p]$ of JORDAN, and which exhibits explicitly the correspondence† between the substitutions of the isomorphic groups.

2. We first define a non-homogeneous linear group A_1 on $2p$ indices which leaves the function $x_1y_1 + \cdots + x_py_p$ invariant modulo 2 and which is holoedrically isomorphic with the abelian group A. To the generators M_i, L_i, N_{ij} of A we make correspond the respective substitutions of A_1:

$$\mu_i: \qquad x_i' = y_i, \qquad y_i' = x_i;$$

$$\lambda_i: \qquad x_i' = x_i + y_i + 1;$$

$$\nu_{ij}: \qquad x_i' = x_i + y_j, \qquad x_j' = x_j + y_i.$$

Then to the general substitution of A,

$$S: \qquad x_i' = \sum_{i=1}^{p}(\alpha_{ij}x_j + \gamma_{ij}y_j), \qquad y_i' = \sum_{j=1}^{p}(\beta_{ij}x_j + \delta_{ij}y_j) \quad (i=1, \cdots, p),$$

will correspond the following substitution of A_1:

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† It is shown in § 6 that this correspondence is in accord with that given by JORDAN.

$$\sigma: \quad \begin{aligned} x_i' &= \sum_{j=1}^{p}(\alpha_{ij}x_j + \gamma_{ij}y_j) + \sum_{j=1}^{p}\alpha_{ij}\gamma_{ij}, \\ y_i' &= \sum_{j=1}^{p}(\beta_{ij}x_j + \delta_{ij}y_j) + \sum_{j=1}^{p}\beta_{ij}\delta_{ij} \end{aligned} \qquad (i=1, \cdots, p).$$

In fact, the general correspondence $S \sim \sigma$ includes the assumed correspondences

$$M_i \sim \mu_i, \qquad L_i \sim \lambda_i, \qquad N_{ij} \sim \nu_{ij} \qquad (i, j=1, \cdots, p).$$

Moreover, if $S_1 \sim \sigma_1$, it is readily verified that

$$M_i S_1 \sim \mu_i \sigma_1, \qquad L_i S_1 \sim \lambda_i \sigma_1, \qquad N_{ij} S_1 \sim \nu_{ij}\sigma_1 \quad (i, j=1, \cdots, p).$$

Since the generators μ_i, λ_i, ν_{ij} leave invariant the function $x_1 y_1 + \cdots + x_p y_p$, the general substitution σ of the group A_1 will leave it invariant.

3. THEOREM.*—*The group A_1 may be represented as a doubly transitive substitution group on the $R_p \equiv 2^{2p-1} - 2^{p-1}$ letters $(x_1 y_1 x_2 y_2 \cdots x_p y_p)$ in which $x_1, y_1, \cdots x_p, y_p$ assume every system of solutions, not all zero, of the congruence*

(1) $$x_1 y_1 + x_2 y_2 + \cdots + x_p y_p \equiv 1 \pmod 2.$$

That A_1 is transitive on the R_p letters may be shown by the usual methods of linear group theory, or directly by the following remark. Let $(\alpha_1 \gamma_1 \cdots \alpha_p \gamma_p)$ be an arbitrary one of the letters. Then $\alpha_1 \gamma_1 + \cdots + \alpha_p \gamma_p \equiv 1 \pmod 2$. One substitution which belongs to A_1 and which replaces $(11\ 00 \cdots 00)$ by $(\alpha_1 \gamma_1 \cdots \alpha_p \gamma_p)$ is the following:

$$\begin{aligned} x_1' &= (\alpha_1\gamma_1 + \alpha_1 + \gamma_1)x_1 + (\alpha_1 + 1)y_1 + \sum_{i=2}^{p}\{(\alpha_1+1)\gamma_i x_i + (\alpha_1+1)\alpha_i y_i\} \\ &\qquad + (\alpha_1+1)(\gamma_1+1), \\ y_1' &= (\gamma_1+1)x_1 + (\alpha_1\gamma_1 + \alpha_1 + \gamma_1)y_1 + \sum_{i=2}^{p}\{(\gamma_1+1)\gamma_i x_i + (\gamma_1+1)\alpha_i y_i\} \\ &\qquad + (\alpha_1+1)(\gamma_1+1), \\ x_j' &= \alpha_j(\gamma_1+1)x_1 + \alpha_j(\alpha_1+1)y_1 + (\alpha_j\gamma_j+1)x_j + \alpha_j y_j \\ &\qquad + \sum\nolimits' (\alpha_j\gamma_i x_i + \alpha_j\alpha_i y_i) + \alpha_j(\alpha_1+\gamma_1+1), \\ y_j' &= \gamma_j(\gamma_1+1)x_1 + \gamma_j(\alpha_1+1)y_1 + \gamma_j x_j + (\alpha_j\gamma_j+1)y_j \\ &\qquad + \sum\nolimits' (\gamma_j\gamma_i x_i + \gamma_j\alpha_i y_i) + \gamma_j(\alpha_1+\gamma_1+1), \end{aligned}$$

where $\sum'$ denotes the summation $i = 2, \cdots, p;\ i \neq j$.

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* For other applications one might employ the theorem that the group A_1 permutes transitively the 2^{2p} functions $a_1 x_1 + b_1 y_1 + \cdots + a_p x_p + b_p y_p + a_1 b_1 + \cdots + a_p b_p$.

To prove that the group is doubly transitive, it now suffices to show that the subgroup leaving the letter $(11\ 00 \cdots 00)$ fixed is transitive on the remaining letters. The conditions that the general substitution σ of A_1 shall leave fixed the letter $(11\ 00 \cdots 00)$ are*

$$\alpha_{i1} + \gamma_{i1} + \sum_{j=1}^{p} \alpha_{ij}\gamma_{ij} \equiv \epsilon_{i1}, \qquad \beta_{i1} + \delta_{i1} + \sum_{j=1}^{p} \beta_{ij}\delta_{ij} \equiv \epsilon_{i1} \quad (i=1, \cdots, p).$$

With these conditions satisfied, S belongs † to the second hypoabelian group (with x_1 and y_1 playing the special rôle). Employing these conditions, we may give to σ the form:

$$x'_i = \sum_{j=1}^{p} (\alpha_{ij}x_j + \gamma_{ij}y_j) + \alpha_{i1} + \gamma_{i1} + \epsilon_{i1},$$
$$y'_i = \sum_{j=1}^{p} (\beta_{ij}x_j + \delta_{ij}y_j) + \beta_{i1} + \delta_{i1} + \epsilon_{i1} \qquad (i=1, \cdots, p).$$

It replaces $(11\ 10 \cdots 00)$ by $(a_1c_1\ a_2c_2 \cdots a_pc_p)$, where

$$a_1c_1 + \cdots + a_pc_p \equiv 1 \pmod 2,$$

if

(2) $$\alpha_{i2} + \epsilon_{i1} = a_i, \qquad \beta_{i2} + \epsilon_{i1} = c_i \qquad (i=1, \cdots, p).$$

To show that the second hypoabelian group contains a substitution S whose coefficients satisfy the conditions (2), we note that the inverse S^{-1} is obtained by replacing α_{ij}, β_{ij}, γ_{ij}, δ_{ij} by δ_{ji}, β_{ji}, γ_{ji}, α_{ji}, respectively, so that the conditions (2) give the following conditions on S^{-1}:

$$\delta_{2i} \equiv a_i + \epsilon_{i1}, \qquad \beta_{2i} \equiv c_i + \epsilon_{i1} \pmod 2 \qquad (i=1, \cdots, p).$$

Hence the coefficients of y'_2 in S^{-1} are fully determined. Also

$$\beta_{21} + \delta_{21} + \sum_{i=1}^{p} \beta_{2i}\delta_{2i} \equiv a_1 + c_1 + (a_1+1)(c_1+1) + \sum_{i=2}^{p} a_ic_i$$
$$\equiv \sum_{i=1}^{p} a_ic_i + 1 \equiv 0 \pmod 2.$$

But ‡ the second hypoabelian group contains such a substitution S^{-1}.

4. Theorem.—*The groups H and A_1 are identical.*

It is first shown that every substitution of A_1 belongs to H. By § 4 of the former paper, μ_i and ν_{ij} (which have the same form as M_i and N_{ij}, respectively)

* Henceforth ϵ_{ij} denotes 1 if $i=j$, but denotes 0 if $i \neq j$.

† Bulletin of the American Mathematical Society, vol. 4 (1898), p. 504.

‡ Dickson, *Linear Groups*, p. 202; or, American Journal of Mathematics, vol. 21 (1899), p. 227.

leave the functions $\phi_3, \phi_4, \phi_5, \cdots$ invariant. Next, λ_1 replaces the general term of ϕ_4 by

$$(x_1' + y_1' + 1 \;\; y_1' \cdots)(x_1'' + y_1'' + 1 \;\; y_1'' \cdots)(x_1''' + y_1''' + 1 \;\; y_1''' \cdots)$$
$$(x_1' + x_1'' + x_1''' + y_1' + y_1'' + y_1''' + 1 \;\; y_1' + y_1'' + y_1''' \cdots),$$

which is seen to be a term of ϕ_4. In like manner, it may be shown that λ_1 leaves invariant $\phi_6, \phi_8, \cdots$; but alters $\phi_3, \phi_5, \cdots$.

It is next shown that every substitution of H belongs to A_1. Let L be an arbitrary substitution of H and let it replace the letters

$$l_1 \equiv (00 \;\; 11 \;\; 00 \;\; \cdots \;\; 00), \quad l_2 \equiv (10 \;\; 11 \;\; 00 \;\; \cdots \;\; 00)$$

by certain letters l_1', l_2', respectively. By § 3, A_1 contains a substitution L' which replaces l_1 by l_1' and l_2 by l_2'. Hence $M \equiv L'^{-1} L$ will belong to H and will leave fixed the letters l_1, l_2. Since M does not alter ϕ_4, it will leave invariant the sum ψ of those terms of ϕ_4 which contain the factor $l_1 l_2$. The general term of ψ is therefore

$$l_1 l_2 (x_1 y_1 \;\; x_2 y_2 \;\; x_3 y_3 \;\; \cdots)(x_1 + 1 \;\; y_1 \;\; x_2 y_2 \;\; x_3 y_3 \;\; \cdots).$$

In view of (1), the last two expressions denote letters if, and only if,

$$\sum_{i=1}^{p} x_i y_i \equiv 1, \qquad y_1 \equiv 0 \pmod 2.$$

But the letters l_1 and l_2 satisfy these congruences. Hence ψ involves exactly $2R_{p-1}$ letters. Henee M must permute amongst themselves the remaining $R_p - 2R_{p-1} \equiv 2^{2p-2}$ letters, the general one of which is

$$(3) \qquad (x_1 \;\; 1 \;\; x_2 y_2 \;\; x_3 y_3 \;\; \cdots), \qquad x_1 + \sum_{i=2}^{p} x_i y_i \equiv 1 \pmod 2.$$

The substitutions of A_1 which leave unaltered the letters l_1 and l_2 permute transitively the 2^{2p-2} letters (3).

Indeed, by § 3, the substitutions of A_1 which leave l_1 fixed have the form

$$\begin{aligned} x_i' &= \sum_{j=1}^{p} (\alpha_{ij} x_j + \gamma_{ij} y_j) + \alpha_{i2} + \gamma_{i2} + \epsilon_{i2}, \\ y_i' &= \sum_{j=1}^{p} (\beta_{ij} x_j + \delta_{ij} y_j) + \beta_{i2} + \delta_{i2} + \epsilon_{i2} \end{aligned} \qquad (i = 1, \cdots, p).$$

The latter leaves l_2 fixed if, and only if,

$$\alpha_{11} = 1, \quad \alpha_{21} = 0, \quad \beta_{11} = 0, \quad \beta_{21} = 0, \quad \alpha_{i1} = \beta_{i1} = 0 \qquad (i = 3, \cdots, p).$$

Let σ_1 denote the general substitution so defined and let S_1 denote the corresponding homogeneous substitution. Let $(c_1 1\ c_2 d_2\ c_3 d_3 \cdots)$ be an arbitrary letter of the form (3). The conditions that σ_1 shall replace $(01\ 11\ 00 \cdots 00)$ by $(c_1 1\ c_2 d_2 \cdots)$ are

$$(4)\quad c_1 = \gamma_{11},\ 1 = \delta_{11},\ c_2 = \gamma_{21} + 1,\ d_2 = \delta_{21} + 1,\ c_i = \gamma_{i1},\ d_i = \delta_{i1}\ \ (i = 3, \cdots, p).$$

To prove that there exists a substitution σ_1 satisfying the conditions (4) we follow the method used at the end of § 3. We observe that S_1^{-1} is the most general substitution of the second hypoabelian group (with x_2, y_2 playing the special rôle) which leaves the index y_1 unaltered. The conditions (4) give the following conditions modulo 2 on S_1^{-1}

$$\alpha_{11} \equiv 1,\ \gamma_{11} \equiv c_1,\ \alpha_{12} \equiv d_2 + 1,\ \gamma_{12} \equiv c_2 + 1,\ \alpha_{1i} \equiv d_i,\ \gamma_{1i} \equiv c_i\ \ (i = 3, \cdots, p).$$

Hence the coefficients of x_1' in S_1^{-1} are fully determined. Also, by (3),

$$\alpha_{12} + \gamma_{12} + \sum_{i=1}^{p} \alpha_{1i}\gamma_{1i} \equiv 1 + c_1 + \sum_{i=2}^{p} c_i d_i \equiv 0 \pmod{2}.$$

But the second hypoabelian group contains a substitution of the form

$$y_1' = y_1,\quad x_1' = \sum_{i=1}^{p} (\alpha_{1i} x_i + \gamma_{1i} y_i),\ \cdots,\quad \left(\alpha_{12} + \gamma_{12} + \sum_{i=1}^{p} \alpha_{1i}\gamma_{1i} \equiv 0,\ \alpha_{11} \equiv 1\right).$$

Next, let M replace $l_3 \equiv (01\ 11\ 00\ \cdots\ 00)$ by a letter l_3' of the form (3). By the preceding result, A_1 contains a substitution T which replaces l_3 by l_3'. Hence $M = TQ$, where Q is a substitution of H which leaves fixed the letters l_1, l_2, l_3. By § 9 of the former paper, Q permutes amongst themselves the R_{p-1} letters $(00\ x_2 y_2\ x_3 y_3 \cdots)$. The theorem may now be established by induction from $p - 1$ to p. We proceed as in § 10 of the earlier paper,* deleting the functions ϕ_3 and $\phi_3^{(p-1)}$. As a basis for the induction, we show that the theorem is true for $p = 2$, whence $R_p = 6$. The six letters

$$(0011),\quad (1011),\quad (0111),\quad (1100),\quad (1101),\quad (1110),$$

cannot be combined to give a term of ϕ_4, so that the latter does not exist when $p = 2$. Evidently ϕ_6 is the product of the six letters. Hence H is the symmetric group on six letters. But the order of the quaternary abelian group modulo 2 is $(2^4 - 1)\, 2^3 (2^2 - 1)\, 2 \equiv 6!$ Hence the groups H and A_1 are identical when $p = 2$†.

*One part of the proof by induction was there omitted, viz., the proof for the case $p = 2$, whence $R_2 = 6$. That G_1 and Γ are identical follows from the equality of their orders (see § 11), or more simply since Q is, for $p = 2$, either the identity or else is M_2, permuting (1101) with (1110), and hence is hypoabelian

† For a direct proof of the holoedric isomorphism of the symmetric group on 6 letters and the quaternary abelian group modulo 2, see *Linear Groups*, p. 99.

5. It follows that the order w_p of H satisfies the recursion formula

$$w_p = R_p(R_p - 1)2^{2p-2} \cdot \frac{w_{p-1}}{R_{p-1}} \equiv (2^{2p} - 1)2^{2p-1} \cdot w_{p-1}.$$

Since $w_2 = (2^4 - 1)2^3(2^2 - 1)2$, we derive the result,

$$w_p = (2^{2p} - 1)2^{2p-1}(2^{2p-2} - 1)2^{2p-3} \cdots (2^2 - 1)2.$$

6. To show that the above correspondence of operators of the isomorphic groups H and A is in accord with that obtained by JORDAN, we note that, in view of p. 241 of *Traité des substitutions*,

$$[11\ 00 \cdots 00] \sim M_1, \quad [10\ 00 \cdots 00] \sim L_1, \quad [10\ 10\ 00 \cdots 00] \sim L_2 L_1 N_{12}.$$

Also (*Traité*, p. 230), $[11\ 00 \cdots]$ leaves $(x_1 y_1\ x_2 y_2 \cdots)$ fixed if $x_1 + y_1 \equiv 0 \pmod 2$, but replaces it by $(x_1 + 1\ y_1 + 1\ x_2 y_2 \cdots)$ if $x_1 + y_1 \equiv 1$, and hence may be designated

$$\mu_1: \quad x_1' = y_1, \quad y_1' = x_1.$$

Likewise, $[10\ 00 \cdots]$ leaves $(x_1 y_1\ x_2 y_2 \cdots)$ fixed if $y_1 \equiv 1$, but replaces it by $(x_1 + 1\ y_1\ x_2 y_2 \cdots)$ if $y_1 \equiv 0$, and hence may be designated

$$\lambda_1: \quad x_1' = x_1 + y_1 + 1.$$

Next, $[10\ 10\ 00 \cdots]$ leaves $(x_1 y_1\ x_2 y_2 \cdots)$ fixed if $y_1 + y_2 \equiv 1$, but replaces it by $(x_1 + 1\ y_1\ x_2 + 1\ y_2\ x_3 y_3 \cdots)$ if $y_1 + y_2 \equiv 0$, and hence may be designated

$$\lambda_2 \lambda_1 \nu_{12}: \quad x_1' = x_1 + y_1 + y_2 + 1, \quad x_2' = x_2 + y_1 + y_2 + 1.$$

It follows that $N_{12} \sim \nu_{12}$. In view of the symmetry, $N_{ij} \sim \nu_{ij}$, etc.

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